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COHOMOLOGY OF STEP FUNCTIONS UNDER IRRATIONAL ROTATIONS

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ABSTRACT

This paper studies the solvability of the functional equation $g(x+\theta)=\lambda f(x)g(x)$, given an irrational θ and a step function f mapping $\mathbf{R}/Z$ (with Lebesgue measure) to the unit circle. Results are applied to find parameterized families of representations of non-regular semi-direct product groups and to display irregularities in the uniform distribution of the sequence $Z\theta$.

1. Introduction

In studying an irrational rotation of the circle, whether from the point of view of number theory, ergodic theory, or representation theory, one is led repeatedly to the following question: Under what conditions on an irrational θ and a function f mapping $\mathbf{R}/Z$ to the unitary group $U(\mathbf{C}^m)$, can the functional equation

$$g(x+\theta)=f(x)g(x)$$

be solved (with nonzero measurable g). Although in this generality the question has proven intractable, work due to H. Furstenberg, W. A. Veech and others has shown that by restricting to special classes of f's, partial answers can be found which give insights into the above fields. In this paper, we restrict to the case of Lebesgue measurability, dimension $m=1$, and f a step function. In so doing, we obtain results which give new interesting examples in each of the above fields and also develop techniques which we hope will be useful in the more general study of the question.

We will need the following terminology. Let $X=\mathbf{R}/Z$ be identified with the unit interval with addition mod 1. Fix an irrational θ in X. Given a real valued function v on X, we build an additive cocycle v on $X\times Z$ by the formula:

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$$v(x,n)=\begin{cases}\sum_{j=0}^{n-1} v(x+j\theta) & \text{for } n>0,\\ 0 & \text{for } n=0,\\ -\sum_{j=1}^{n} v(x-j\theta) & \text{for } n<0.\end{cases}$$

Then v satisfies the additive cocycle identity:

$$v(x,n)+v(x+n\theta,k)=v(x,n+k).$$

We call v a coboundary if there is a measurable real valued function w such that $v(x)=w(x+\theta)-w(x)$; we say that v_1 is cohomologous to v_2 if their difference is a coboundary.

If we let $f(x)=\exp(2\pi i v(x))$, then f maps X to the unit circle, and $f(x,n)=\exp(2\pi i v(x,n))$ satisfies the multiplicative cocycle identity:

$$f(x,n)f(x+n\theta,k)=f(x,n+k).$$

Thus f is a multiplicative cocycle. It is a coboundary if there is a measurable unit circle valued function g such that $f(x)=g(x+\theta)/g(x)$; f_1 is cohomologous to f_2 if their quotient is a coboundary. A result due to C. Moore and K. Schmidt [10] implies that v is an additive coboundary (with respect to Borel measure) if and only if $\exp(2\pi i s v)$ is a multiplicative coboundary (Borel) for every real s. We will later see examples of v's which are not additive coboundaries but with $\exp(2\pi i s v)$ a multiplicative coboundary for some real s's.

Both additive and mutliplicative cocycles which come from step functions are useful in answering number theoretic questions about uniform distribution. The Kronecker–Weyl theorm states that the multiples of θ are uniformly distributed mod 1 in the sense that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_I(j\theta)=\mu(I)$$

where I is any interval, χ_I is its characteristic function, and μ is Lebesgue measure. If for t in X, we let $v(x)=\chi_{[0,t)}(x)-t$, then the Kronecker–Weyl theorem says that the corresponding additive cocycle has the property that for each x, $v(x,n)/n\to 0$ as $n\to\infty$. A natural question to ask is whether the $v(x,n)$'s are bounded in n for each x. This is equivalent to asking whether v is a coboundary with a bounded w. H. Kesten [7] and Karl Petersen [11] showed that this is the case if and only if t is a multiple of θ.

A slightly more subtle question looks for regularity in the sequence $\Sigma_{j=0}^{n-1}\chi_{[0,t)}(x+j\theta)$. For example, one might ask whether this sequence is uniformly distributed mod q, $q>1$ (i.e., as $n\to\infty$, does it tend toward being equally proportioned among the congruence classes mod q). An easy application of the ergodic theorem shows that for almost all x this sequence will fail to be uniformly distributed mod q, if the function f defined by $f(x) = \exp(2\pi i \chi_{[0,t)}/q)$ is a multiplicative coboundary with a g whose integral is nonzero. Exponentiating the Petersen result and using the information in Corollary 2.3 of this paper, it is easy to produce examples of t of the form $k\theta$ which have $\Sigma_{j=0}^{n-1} \chi_{[0,t)}(x + j\theta)$ not uniformly distributed mod q for some q. Veech [14] and Stewart [13] show that in the case of θ with bounded partial quotients in its continued fraction expansion, only t's which are multiples of θ can give sequences which fail to be uniformly distributed mod q for some q. What is surprising is that for θ with unbounded partial quotients, Veech and Stewart produce a class of t's which are not multiples of θ yet such that $\Sigma_{j=0}^{n-1} \chi_{[0,t)}(x + j\theta)$ fails to be uniformly distributed mod q for some q. In so doing, they find multiplicative coboundaries of the form $\exp(2\pi i \chi_{[0,t)}/q)$ such that (by the result of Schmidt and Moore) $\chi_{[0,t)}$ is not an additive coboundary. In Section 2 we extend these results of Veech and Stewart by replacing $1/q$ with an irrational.

Section 3 contains the major results of this paper. In it we use step functions with more than two points of discontinuity to explore further the relationship between additive and multiplicative coboundaries and questions of uniform distribution. Let $v = \chi_{[0,t)} - \chi_{[r,r+t)}$. Furstenberg, Keynes, and Shapiro [4] show that the corresponding additive cocycle, $v(x, n)$, is bounded in n for some x if and only if either r or t is a multiple of θ. In Section 3, we explore the more complicated behavior of the corresponding multiplicative cocycle. Let $s \in R$, and $f = \exp(2\pi i s v)$, with v as above. Theorem 3.3 produces an uncountable collection of pairs r and t which are not multiples of θ and yet for which f is a multiplicative coboundary. At first glance this seems not surprising in view of the results of Veech and Stewart in the single interval case. However, there are two major distinctions: Theorem 3.3 is valid for all θ, not just θ with bounded partial quotients. Also, Theorem 3.3 is valid for all s. This means (by the result of Moore and Schmidt) that in this case the additive cocycle $v = \chi_{[0,t)} - \chi_{[r,r+t)}$ is itself an additive coboundary. Thus we produce an example of an additive coboundary which is bounded in x but which is not the coboundary of a bounded w. In terms of uniform distribution, an uncountability argument gives an example of r and t such that the corresponding sequence of $v(x, n)$'s is unbounded yet not uniformly distributed mod q for any q.

Theorem 3.3 is also related to some results of Veech [16] concerning skew products. If f maps X into a group G, we define the skew product $S_{\theta,f}$ mapping $X \times G$ into itself by $S_{\theta,f}(x, \gamma) = (x + \theta, f(x)\gamma)$. For G compact and abelian, an application of the Peter–Weyl theorem shows that $S_{\theta,f}$ is ergodic if and only if $\chi \circ f$ is not a (multiplicative) coboundary for any nontrivial character χ of G [16]. The abelian case of Veech's result is that for G a finite group and θ with bounded partial quotients, $S_{\theta,f}$ is ergodic when f is a step function with rational points of discontinuity. Theorem 3.3 shows that the condition that f have rational points of discontinuity cannot be removed entirely, even for finite G and θ with bounded partial quotients. The other major result of Section 3, Theorem 3.1, extends the abelian case of Veech's result by giving one set of conditions under which the requirement of rational points of discontinuity can be removed.

Although historically, the questions raised in this paper have been derived from questions in ergodic theory and uniform distribution, the answers found here open up applications to the representation theory of non-type I groups as well. An explicit construction ([9], [12]) builds representations of non-type I groups which have type I normal subgroups, from cocycles; the equivalence of the representations corresponds to the cohomology of the cocycles. All 1-dimensional cocycles such as the ones studied in this paper give irreducible representations by this construction; for higher dimensional cocycles the irreducibility is not automatic but still can be determined from the cocycle alone. Further, by varying the measure class and dimension, this construction gives all the irreducible unitary representations of these groups. In many of the interesting examples, (e.g. the Mautner group, the discrete Heisenberg group), the cocycles which occur are cocycles of an irrational rotation. Many other cases can be reduced to the irrational rotation case using general theory (see [2]). Thus an understanding of cocycles of an irrational rotation would reveal the representation theory of large class of groups previously thought to be inaccessible.

The results mentioned above given some progress in this direction. One of the consequences of Theorem 3.1 is to give a complete description of the cohomology of two point step functions in the case of θ with bounded partial quotients. This gives an uncountable parametrized family of cocycles which in turn give a family of representations of the Mautner group. In [1], the results of this paper are used as well to extend the known families of representations [2], [6] which were based on exponential functions and higher dimensional cocycles containing them. There is hope that these richer families of representations can be used to explore questions like mutually singular Plancherel measures. In addition they reveal a previously unknown dependence of the representation theory of these groups on the continued fraction expansion of the irrational θ from which they are constructed [1].

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2. Step functions with two points of discontinuity

When there is no ambiguity we think of arbitrary real numbers as elements of X by identifying them with their congruence class mod 1. When more care is needed we use the following notation:

$[x]$ = the greatest integer in x;
$\|x\| = \min\{|j-x| : j \in Z\}$;
$\{x\} = x - [x]$;
$\underset{\sim}{x} = x$ minus the closest integer to x (with $\underline{j+\frac{1}{2}} = \frac{1}{2}$).

Finally, for x in X or $\mathbf{R}$, define $e(x) = \exp(2\pi i x)$.

Let t and $s \in X$. We begin with the question of which functions of the form $f_{t,s} \equiv e(s\chi_{[1-t,1)})$ are multiplicative coboundaries (since translates of coboundaries are coboundaries, we arbitrarily fix the location of the interval). We study this by asking the slightly more general question of which $f_{t,s}$'s are multiples of coboundaries. That is, we look for the existence of Lebesgue measurable functions g and constants λ, with $|g| \equiv 1$ and $|\lambda| = 1$ such that

$$g(x+\theta) = \lambda f_{t,s}(x) g(x). \tag{1}$$

Theorem 2.2, its corollary and Theorem 2.5 exploit some limited continuity among coboundaries to produce solutions to (1); Theorems 2.4 and 2.6 are nonexistence results first proved by W. A. Veech. Taken together these theorems give an almost complete answer to the question of when (1) can be solved.

This answer involves number theoretic relationships between θ, t, and s which are expressed in terms of the continued fraction expansion of θ. We give an outline of the facts we will need; for more detail see [5] or [8]. Let θ have continued fraction expansion $[a_1, a_2, \ldots]$. Then the a_i are called the partial quotients of θ; $[a_1, a_2, \ldots, a_k] = m_k/n_k$ (in lowest terms) are called the convergents; the n_k are called the denominators. The convergents give the best rational approximations to θ relative to the size of denominator; the denominators tell which multiples of θ best approximate integers. How well they approximate integers is limited relative to the rate of growth of the denominators. The following Lemma makes this precise and gives some helpful relationships between the partial quotients and the convergents:

LEMMA 2.1. (i) $\|n_k\theta\| = \min\{\|j\theta\| : 0 < j < n_{k+1}, j \in Z\}$.

(ii) (a) $1/(n_k(n_k + n_{k+1})) < |\theta - m_k/n_k| < 1/(n_k n_{k+1})$ and thus

(b) $\frac{1}{2} < n_{k+1} \| n_k \theta \| < 1$.

(iii) $m_k = a_k m_{k-1} + m_{k-2}$; $n_k = a_k n_{k-1} + n_{k-2}$, *for* $k \geqq 2$. (*We take* $m_0 = 0$, $m_1 = 1$, $n_0 = 1$, $n_1 = a_1$.)

(iv) $a_{k+1} \| n_k \theta \| + \| n_{k+1} \theta \| = \| n_{k-1} \theta \|$.

PROOF. See [5] or [8].

If θ has its a_k's bounded we say θ has bounded partial quotients. For such θ, (iii) implies that n_k / n_{k+1} is bounded away from 0, so that the limitation on how fast integers can be approximated given in (iib) can be strengthened. For θ with bounded partial quotients, we have that the sequence of $n_k \| n_k \theta \|$ is bounded away from 0, and also that the sequence $n_{k-k_0} \| n_k \theta \|$ is bounded away from 0 for any fixed k_0. We will see that this fact makes the question of the solvability of (1) have quite different answers depending on whether θ has bounded partial quotients.

We are ready now for the main results of this section.

THEOREM 2.2. *For any* $s \in X$ *the function* $f_{\theta,s}$ *is a multiple of a coboundary, with solution to* (1) *given by* $\lambda = e(-s\theta)$ *and* $g(x) = e(-sx)$.

PROOF. Clear by substitution. (Recall that arithmetic in the argument of g is mod 1.)

REMARK. This solution is the key to the existence results of this paper; thus we include some remarks about its discovery. The solution was found by replacing θ with a sequence of rational approximations. The equation $g(x + p/q) = \lambda f_{p/q,s}(x) g(x)$ has a solution if $\lambda^q = e(-sp)$. It can be built by partitioning X into q pieces, taking $g(x) = 1$ on the first subinterval, and then using the functional equation to define it on the $(p+1)$st (mod q) subinterval, then the $(2p+1)$st (mod q), etc. The condition on λ insures that this function will match up when it gets back to the first subinterval. The value of this function on the jth subinterval will be $\lambda^h e(sk)$ where h is the number of steps required to get to the jth subinterval in the above procedure and k is the number of those steps, excluding the final one, that land in one of the last p subintervals. We see that h and j satisfy $hp + 1 = kq + j$. Taking $\lambda = e(-sp/q)$, the resulting g has the value $e(-s(j-1)/q)$ on the jth subinterval. As $p/q \to \theta$, these solutions approach the solution for θ given in the statement of the theorem.

Since similar solutions are easy to find for any f under rational rotations, this procedure could have wide application. However, the solutions for p/q usually fail to converge as $p/q \to \theta$. This result is a streak of continuity in an essentially discontinuous field (as are most of the existence results in this paper). Much could be gained by a more general understanding of where continuity is likely to occur.

We use the fact that translates of solutions to the functional equation for f give solutions for the translate of f; and that products of solutions for f_1 and solutions for f_2 give solutions for $f_1 f_2$ to obtain the following:

COROLLARY 2.3. *For any $s \in X$ and $k \in Z$, $f_{k\theta,s}$ is a multiple of a coboundary, with solution to* (1) *given by $\lambda = e(-s\{k\theta\})$ and $g(x) = e(-skx)P_k(x)$, where P_k is the following step function:*

For $k > 0$, P_k has discontinuities at $0, -\theta, \dots, -(k-1)\theta$, and takes on the values $e(sj)$, $j = 0, 1, \dots, k-1$ in that order as x goes from 0 to 1.

For $k < 0$, P_k has discontinuities at $0, \theta, \dots, -k\theta$, and takes on the values $e(-sj)$, $j = 0, 1, \dots, -k$ in that order as x goes from 0 to 1.

PROOF. Given a function f on X and a $y \in X$, let ${}^y f$ be defined by

$${}^y f(x) = f(x+y).$$

Using this notation, for $k > 0$,

$$f_{k\theta,s} = e(-s[k\theta])(f_{\theta,s})({}^{\theta}f_{\theta,s})({}^{2\theta}f_{\theta,s})\cdots({}^{(k-1)\theta}f_{\theta,s}).$$

Thus if $g_{\theta,s}$, $\lambda_{\theta,s}$ is the solution for $t = \theta$ given in Theorem 2.2, then

$$g = (g_{\theta,s})({}^{\theta}g_{\theta,s})^{2\theta}g_{\theta,s}\cdots({}^{(k-1)\theta}g_{\theta,s}),$$

and $\lambda = e(s[k\theta])(\lambda_{\theta,s})^k$ give a solution for $t = k\theta$ (mod 1). The corollary for $k > 0$ follows by noting that

$${}^{j\theta}g_{\theta,s}(x) = \begin{cases} e(-s(x+\{j\theta\})) & \text{for } x \in [0, 1-\{j\theta\}) \\ e(-s(x+\{j\theta\}-1)) & \text{for } x \in [1-\{j\theta\}, 1) \end{cases}$$

and that g can be replaced by any multiple of itself and still give a solution. (We replace g by $e(s\sum_{j=1}^{k-1}\{j\theta\})g$.) To get the result for $k < 0$, write $f_{k\theta,s} = e(s)({}^{k\theta}f_{-k\theta,-s})$. Thus $\lambda_{k\theta,s} = e(-s)(\lambda_{-k\theta,-s})$, and $g_{k\theta,s} = {}^{k\theta}g_{-k\theta,-s}$.

REMARK. W. A. Veech [14] proved the special case of this corollary where $s = \frac{1}{2}$ and t is an even multiple of θ. M. Stewart [13] extended Veech's result to the special case where s is rational with denominator q and t is a multiple of $q\theta$. Karl Petersen [11] has a result which proves the existence of the solutions given in the corollary, but which says nothing about the form of the g.

We observe that if g, λ is a solution to (1) then g', λ' is also, where $g'(x) = g(x)e(jx)$ and $\lambda' = \lambda e(j\theta)$, $j \in Z$. (In fact, then all solutions must be of this form since all eigenvalues of the irrational rotation are of the form $e(j\theta)$.) Thus we see that $f_{k\theta,s}$ is an actual coboundary (rather than just a multiple) if $s\{k\theta\} = m + n\theta$ for some integers m and n. So for example, $f_{k\theta,p/q}$ is a coboundary if both k and $[k\theta]$ are divisible by q. For simple examples, such as when $[k\theta] = 0$, it is easy to see that the corresponding $g'_k = P_k$ has nonzero integral. Thus, we have examples of sequences $\sum_{j=0}^{n-1} \chi_{[0,k\theta)}(x + j\theta)$ which are not uniformly distributed mod q.

With irrational s, we have examples of coboundaries such as $f_{\theta,1/\theta}$. Again for simple examples such as this we easily see that $\int g d\mu \neq 0$. Analogously to the rational case, we thus produce examples of sequences of the form $s\sum_{j=0}^{n-1} \chi_{[0,t)}(x + j\theta)$ which are not uniformly distributed mod 1.

The next result, which was first proved by W. A. Veech [15], shows that when θ has bounded partial quotients, (1) has no measurable solution except under the conditions of the corollary. Thus, in the bounded partial quotients case, we have found all the coboundaries (and multiples) among the $f_{t,s}$'s, and thus all possible examples of failure of uniform distribution. The proof given here is a simplication of the proof in [15], but it uses essentially the same technique. As we will use this technique in all the nonexistence proofs in this paper, we first outline its basic principles.

Given an f mapping X to the unit circle and an integer $n > 0$, we define the function $f^{(n)}$ by

$$f^{(n)}(x) = f(x)f(x + \theta)f(x + 2\theta)\cdots f(x + (n-1)\theta).$$

(This is just the cocycle $f(x, n)$ defined in the introduction.) If f is a multiple of a coboundary then

$$f^{(n)}(x) = \lambda^{-n} g(x + n\theta)/g(x)$$

for some measurable g and constant λ, $|\lambda| = 1$. Thus if we let

$$c^{(n)} = \int f^{(n)}(x)\, d\mu(x),$$

we have by the continuity of translation in L^2 that

$$\lim_{\|n\theta\| \to 0} |c^{(n)}| = 1. \tag{2}$$

But then $(f^{(n)} - c^{(n)}) \to 0$ in L^2 as $\|n\theta\| \to 0$ (since the Fourier series map is an isometry). So if we fix an ε and let

$$A_\varepsilon^{(n)} = \{x : |f^{(n)}(x) - c^{(n)}| < \varepsilon\},$$

we have that for each $\varepsilon > 0$,

$$\lim_{\|n\theta\| \to 0} \mu(A_\varepsilon^{(n)}) = 1. \tag{3}$$

Recall that if f has discontinuities at $t_1, t_2, \dots, t_m$ then $f^{(n)}$ has discontinuities at $t_i - j\theta$ for $1 \leqq i \leqq m$ and $0 \leqq j < n$. If the value of f jumps by $e(s_i)$ at t_i then the value of $f^{(n)}$ jumps by $e(s_i)$ at each of the points $t_i - j\theta$ (unless this point happens also to be a $t_{i'} - j'\theta$; in this case the total jump is the sum of the jumps contributed by the points which coincide). We think of m columns of discontinuities, each with n points in it. If $t_i - t_{i'} = p\theta$, then the ith and i'th columns overlap with p points in each column that don't coincide with points in the other column. The intervals between discontinuities in $f^{(n)}$ can be thought of as pairings of these points of discontinuity. To apply the remarks above, we take ε to be $\frac{1}{2}$ the minimum absolute value of the ratios of successive values of f, so that no two adjacent intervals between discontinuities of $f^{(n)}$ can both belong to $A_\varepsilon^{(n)}$. Then if we let $\delta^{(n)}$ be the length of the smallest interval between discontinuities of $f^{(n)}$ and $d^{(n)}$ be the number of discontinuities, (3) shows that

$$\lim_{\|n\theta\| \to 0} d^{(n)}\delta^{(n)} = 0. \tag{4}$$

We use estimates and counting arguments on the possible pairings to show that for certain f's this cannot happen.

Now we are ready to state the theorem:

THEOREM 2.4. *If θ has bounded partial quotients and $t \notin Z\theta$, then $f_{t,s}$ is not a multiple of a coboundary for any nonzero s.*

PROOF. Let θ have bounded partial quotients, with convergents m_k/n_k; and let $t \notin Z\theta$. Suppose there is a measurable g and a λ such that $g(x+\theta) = \lambda f_{t,s}(x) g(x)$. Then we have as in (2) that

$$\left| \int f_{t,s}^{(n_k)}(x) d\mu(x) \right| = |c^{(n_k)}| \to 1, \qquad \text{as } k \to \infty.$$

With $\varepsilon = \frac{1}{2}|1 - e(s)|$, we have by (3) that $\lim \mu(A_\varepsilon^{(n_k)}) = 1$, and by (4) that $\lim_{k\to\infty} d^{(n_k)}\delta^{(n_k)} = 0$.

Now, the $2n_k$ points of discontinuity of $f_{t,s}^{(n_k)}$ are distinct since $t \notin Z\theta$, so that

$d^{(n_k)} = 2n_k$. Thus we have

$$\lim_{k\to\infty} n_k \delta^{(n_k)} = 0. \tag{5}$$

The intervals of constant value of $f_{t,s}^{(n_k)}$ are all of length $\|p\theta\|$ or $\|t-p\theta\|$ with $p \in Z$, $|p| < n_k$. Recall that by Lemma 2.1,

$$\min_{0<p<n_k} \|p\theta\| = \|n_{k-1}\theta\|,$$

and $n_k \|n_{k-1}\theta\|$ is bounded away from 0. Thus (5) implies that $\delta^{(n_k)}$ is eventually of the form $\|t-p\theta\|$.

Let p_{n_k} be such that

$$\delta^{(n_k)} = \|t - p_{n_k}\theta\| = \min_{|p|<n_k} \|t-p\theta\|.$$

Then

$$\lim_{k\to\infty} n_k \|t - p_{n_k}\theta\| = 0$$

and also

$$n_k \|t - p_{n_{k+1}}\theta\| < n_{k+1}\|t - p_{n_{k+1}}\theta\| \to 0 \qquad \text{as } k\to\infty,$$

and thus

$$\lim_{k\to\infty} n_k \|p_{n_k}\theta - p_{n_{k+1}}\theta\| = 0.$$

But since $|p_{n_k} - p_{n_{k+1}}| < n_k + n_{k+1} \leqq n_{k+2}$, we have by Lemma 2.1 that $\|p_{n_k}\theta - p_{n_{k+1}}\theta\| \geqq \|n_{k+2}\theta\|$, unless $p_{n_k} = p_{n_{k+1}}$. Therefore since $n_k\|n_{k+2}\theta\|$ is bounded away from 0, p_{n_k} is eventually constant. Then t is a multiple of θ, which contradicts our assumption.

The next two theorems show that conditions determining the solvability of (1) are more complicated when θ has unbounded partial quotients. They contain the bounded partial quotient results as special cases.

THEOREM 2.5. *Let* $t = \sum_{k=0}^{\infty} b_k n_k \theta \pmod 1$, *where* $b_k \in Z$. *Suppose* $\sum_{k=0}^{\infty} |b_k|\, n_k \|n_k\theta\| < \infty$ *and also* $\sum_{k=0}^{\infty}\|b_k s\| < \infty$. *Then* $f_{t,s}$ *is a multiple of a coboundary.*

PROOF. The condition that $\sum_{k=0}^{\infty} |b_k|\, n_k \|n_k\theta\| < \infty$ implies that $\lim_{k\to\infty} b_k n_k \|n_k\theta\| = 0$, and thus by Lemma 2.1 that $|b_k| < a_{k+1}$ for k sufficiently large. Choose an integer $k_0 > 0$ such that $|b_k| < a_{k+1}$ for $k \geqq k_0$. Define $t_k = \sum_{j=0}^{k} b_j n_j \theta$, and for $k \geqq k_0$, let

$$f_k = \begin{cases} ({}^{t_{k-1}} f_{b_k n_k \theta, s}) & \text{if } b_k n_k \theta \in [0, \tfrac{1}{2}), \\ e(-s)({}^{t_{k-1}} f_{b_k n_k \theta, s}) & \text{if } b_k n_k \theta \in [\tfrac{1}{2}, 1). \end{cases}$$

Then $f_{t,s} = (f_{t_{k_0-1},s}) \prod_{k=k_0}^{\infty} f_k$, with the product converging in L^1. Thus it will suffice to find solutions λ_k, g_k for f_k such that $\prod_{k=k_0}^{\infty} g_k$ converges in L^1. Since $|g_k| \equiv 1$, we can show $\{\prod_{k=k_0}^{m} g_k\}_{m=k_0}^{\infty}$ is Cauchy by showing that $\sum_{k=k_0}^{\infty} \|1 - g_k\|_1 < \infty$. This is possible with a careful choice of g_k. Using the remark following Corollary 2.3 (with $j = n_k$ times the closest integer to $b_k s$), and ignoring translation (since $\|1 - g_k\|_1$ is translation invariant), we can alter the solution for f_k given in Corollary 2.3 and take $g_k(x) = e(-n_k \underline{b_k} s x) P_{b_k n_k}(x)$.

Since the solution for $f_{-b_k n_k \theta, s}$ is just a translate of the conjugate of the solution for $f_{b_k n_k \theta, s}$, we can assume without loss of generality that b_k is positive. Recall that $P_{b_k n_k}$ has discontinuities at $0, -\theta, \ldots, -(b_k n_k - 1)\theta$. We will need some information about how these points fall in the unit interval. Let the points $0, -\theta, \ldots, -(n_k - 1)\theta$ be ordered from 0 to 1 by $0 = j_0 \theta < j_1 \theta < \cdots < j_{n_k - 1}\theta < 1$. Call these the primary points. The next multiple, $-n_k \theta$, will be a distance $\|n_k \theta\|$ from 0. Without loss of generality, suppose it is to the left of 0, that is in $[\frac{1}{2}, 1)$. As we continue to mark off negative multiples of θ up through $-(2n_k - 1)\theta$ we will get points $\|n_k \theta\|$ to the left of each of the primary points. Call these the secondary points. Since $\min_{0<j<n_k} \|j\theta\| > \|n_k \theta\|$, these two sets of points will interlace. If $b_k > 2$, continue to mark off the multiples from $-2n_k \theta$ through $-(3n_k - 1)\theta$ to get points $\|n_k \theta\|$ to the left of each of the secondary points, etc., until we have marked off the b_k-ary points. Since (by Lemma 2.1)

$$\min_{0<j<n_k} \|j\theta\| = \|n_{k-1}\theta\| > a_{k+1} \|n_k \theta\| \geqq b_k \|n_k \theta\|,$$

the b_k sets of points will interlace. Thus the distance between discontinuities of $P_{b_k n_k}$ have the following repeating pattern (starting at 0): one of unknown length followed by $(b_k - 1)$ distances of length $\|n_k \theta\|$.

Now we are ready to show $\sum_{k=k_0}^{\infty} \|1 - g_k\|_1 < \infty$. Because $\sum_{k=0}^{\infty} |b_k| n_k \|n_k \theta\| < \infty$, we can change the value of g_k on the $(b_k - 1)n_k$ pieces of length $\|n_k \theta\|$ without affecting the convergence of $\sum_{k=k_0}^{\infty} \|1 - g_k\|_1$. Thus we can replace $P_{b_k n_k}$ by a step function $P_{b_k n_k}^{(0)}$ which has discontinuities only at $0 = j_0 \theta < j_1 \theta < \cdots < j_{n_k - 1}\theta < 1$ and which takes on the value $e(b_k s i)$ on $[j_i \theta, j_{i+1}\theta)$. Now to make the estimates simpler we would like to replace $P_{b_k n_k}^{(0)}$ with a step function $P_{b_k n_k}^{(n_k - 1)}$ which takes on the same values in the same order but has its discontinuities at $0, 1/n_k, \ldots, (n_k - 1)/n_k$. For i between 0 and $n_k - 1$, define $P^{(i)}_{b_k n_k}$ to have discontinuities at $0, 1/n_k, \ldots, i/n_k, j_{i+1}\theta, \ldots, j_{n_k-1}\theta$; and take on the values $e(mb_k s)$, $m = 0, 1, \ldots, n_k - 1$ in that order as x goes from 0 to 1. Then

$$\sum_{k=k_0}^{\infty} \|P^{(0)}_{b_k n_k} - P^{(n_k-1)}_{b_k n_k}\|_1 \leqq \sum_{k=k_0}^{\infty} \sum_{i=1}^{n_k-1} \|P^{(i-1)}_{b_k n_k} - P^{(i)}_{b_k n_k}\|_1$$
$$= \sum_{k=k_0}^{\infty} |1 - e(b_k s)| \sum_{i=1}^{n_k-1} \|i/n_k - j_i\theta\|.$$

Now, $|1 - e(b_k s)| = 2|\sin(\pi\|b_k s\|)| < 2\pi\|b_k s\|$. Also, by noting that the order of multiples of m_k/n_k is the same as the order of multiples of θ in $[0,1)$ (since $\|j\theta - jm_k/n_k\| < 1/n_{k+1}$ while $\|j_i\theta - j_{i+1}\theta\| \geqq \|n_{k-1}\theta\| > 1/n_{k+1}$), we see that

$$\sum_{i=1}^{n_k-1} \|i/n_k - j_i\theta\| = \sum_{i=1}^{n_k-1} \|j_i m_k/n_k - j_i\theta\|$$
$$\leqq \sum_{i=1}^{n_k-1} j_i/n_k n_{k+1}$$
$$< n_k/2n_{k+1}$$
$$< 1.$$

Thus

$$\sum_{k=k_0}^{\infty} \|P^{(0)}_{b_k n_k} - P^{(n_k-1)}_{b_k n_k}\|_1 < 2\pi \sum_{k=0}^{\infty} \|b_k s\| < \infty;$$

and so we may replace $P^{(0)}_{b_k n_k}$ with $P^{(n_k-1)}_{b_k n_k}$ without affecting the convergence of the $\Sigma_{k=k_0}^{\infty}\|1 - g_k\|_1$.

We complete the proof by showing the convergence of $\Sigma_{k=k_0}^{\infty}\|1 - g'_k\|_1$ where $g'_k(x) = e(-n_k \underline{b_k s} x)P^{(n_k-1)}_{b_k n_k}(x)$.

$$\sum_{k=k_0}^{\infty} \|1 - g'_k\|_1 = \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k-1} \int_{i/n_k}^{(i+1)/n_k} |1 - e(-n_k \underline{b_k s} x)P^{(n_k-1)}_{b_k n_k}(x)|\, d\mu(x)$$
$$= \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k-1} \int_{i/n_k}^{(i+1)/n_k} |1 - e(-n_k \underline{b_k s} x + b_k s i)|\, d\mu(x)$$
$$= \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k-1} \int_{i/n_k}^{(i+1)/n_k} |1 - e(-n_k \underline{b_k s} x + \underline{b_k s} i)|\, d\mu(x)$$
$$= 2 \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k-1} \int_{i/n_k}^{(i+1)/n_k} |\sin \pi(\underline{b_k s} i - \underline{b_k s} n_k x)|\, d\mu(x)$$

$$= 2 \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k - 1} \int_{i/n_k}^{(i+1)/n_k} \sin \pi(\| b_k s \| i - \| b_k s \| n_k x) d\mu(x)$$

$$= 2 \sum_{k=k_0}^{\infty} \sum_{i=0}^{n_k - 1} (1 - \cos \pi \| b_k s \|)/(\pi n_k \| b_k s \|)$$

$$\leqq 2\pi \sum_{k=k_0}^{\infty} \| b_k s \| < \infty.$$

REMARK. W. A. Veech [14] and M. Stewart [13] have the following weaker version of the special case of this result where s is rational with denominator q: If $t = \Sigma_{k=0}^{\infty} b_k n'_k q\theta$, where n'_k is a denominator for $q\theta$; and if $\Sigma_{k=0}^{\infty} | b_k | n'_k \| n'_k \theta \| < \infty$, and further $\Sigma_{k=0}^{\infty} \| b_k s \| < \infty$, then (1) has a solution. This differs from the special case of our result in writing t in terms of $q\theta$ instead of θ. It can be deduced from the special case of Theorem 2.5 by noting that if f is a multiple of a coboundary for θ, then it is a multiple of a coboundary for θ/q (rewrite the functional equation $g(x + \theta) = \lambda f(x) g(x)$ as

$$\begin{aligned} &g(x + \theta/q) g(x + 2\theta/q) \cdots g(x + \theta) \\ &\quad = \lambda f(x) g(x) g(x + \theta/q) \cdots g(x + (q-1)\theta/q)). \end{aligned}$$

The following theorem of Veech and Stewart is the corresponding nonexistence result for θ with unbounded partial quotients. Note that unlike the bounded partial quotients result, it does not exclude all t's except those in the existence theorem.

THEOREM 2.6. *If $f_{t,s}$ is a multiple of a coboundary, then t can be written in the form $t = \Sigma_{k=0}^{\infty} b_k n_k \theta$, $b_k \in Z$, where*

$$\lim_{k \to \infty} b_k n_k \| n_k \theta \| = 0 \quad \textit{and} \quad \lim_{k \to \infty} \| b_k s \| = 0.$$

PROOF. See [13], [14].

3. Arbitrary step functions

Any step function with m points of discontinuity is a multiple of a translate of one which has a discontinuity at 0 and takes on 1 as its first value. Such a function can be written uniquely as a product of $m - 1$ functions of the form f_{t_i, s_i} as follows: Let f have discontinuities at $0 < 1 - t_1 < 1 - t_2 < \cdots < 1 - t_{m-1}$; and take on the values 1, $e(s_1), e(s_2), \ldots, e(s_{m-1})$ in that order from 0 to 1. Then

$$f = (f_{t_1, s_1})(f_{t_2, s_2 - s_1}) \cdots (f_{t_{m-1}, s_{m-1} - s_{m-2}}).$$

This representation will enable us to use the results and techniques of the last section to determine some conditions governing which step functions are multiples of coboundaries.

We have already seen that multiples of coboundaries are preserved under products and translations, so we can easily use the positive results of Section 2 to build step functions with m points of discontinuity which are multiples of coboundaries. For θ with bounded partial quotients, all such functions will have the property that at least one of the distances between discontinuities is a multiple of θ (mod 1). One could hope that these are the only step functions f with solutions. We shall see later that this is not true in general; however, the following theorem gives a set of conditions under which this simplest possible situation does hold.

THEOREM 3.1. *Let f be a step function with points of discontinuity at $t_0 < t_1 < \cdots < t_{m-1}$ and with*

$$f(x) = \begin{cases} \gamma_i & \text{for } x \in [t_{i-1}, t_i),\ 1 \leqq i < m, \\ \gamma_m & \text{for } x \in [t_{m-1}, 1) \cup [0, t_0). \end{cases}$$

Suppose at least one of the distances $t_i - t_{i'} \notin Z\theta$, and further that no product of m_0 distinct ratios of the form γ_{i+1}/γ_i is 1 for any $m_0 < m$. Then for θ with bounded partial quotients, f is not a multiple of a coboundary.

PROOF. Without loss of generality, let $t_0 = 0$, and replace t_i by $1 - t_i$. Let $\gamma_{i+1}/\gamma_i = e(s_i)$, $s_i \in X$. There is a solution for f if and only if there is one for f/γ_1. Thus it will suffice to prove the theorem for functions of the form

$$f = f_{t_1, s_1} f_{t_2, s_2} \cdots f_{t_{m-1}, s_{m-1}},$$

under the condition that at least one of the $t_i \notin Z\theta$ and that no partial sum of the s_i's, $\sum_{j=1}^{m_0} s_{i_j}$, $m_0 \leqq m-1$, is an integer.

Suppose there is a solution to $g(x+\theta) = \lambda f(x) g(x)$. As before, we will study the effects the existence of this solution has on the behavior of the sequence of $f^{(n_k)}$. We will replace this sequence by a succession of other sequences in such a way as to preserve certain essential aspects of this behavior, and eventually reach a contradiction.

Accordingly, we rename the sequence $f^{(n_k)} = f^{(0,k)}$. Recall that $f^{(0,k)}$ has jump discontinuities of size $e(s_i)$ at the points $-t_i - p\theta$ for $0 \leqq p < n_k$ where $0 \leqq i < m$, and where we take $t_0 = 0$ and $s_0 = -\sum_{i=1}^{m-1} s_i$. If some of the distances $t_i - t_{i'}$ are in $Z\theta$, then some of these discontinuities may coincide; in that case the total jump at that point is the product of the appropriate $e(s_i)$'s. For each i, $0 \leqq i < m$, we define

$$t_i^{(0,k)} = t_i$$

for all k. Now we describe how to obtain $t_i^{(j+1,k)}$ from $t_i^{(j,k)}$ If $\delta^{(j,k)} =$ the minimum distance between discontinuities of $f^{(j,k)}$ is in $Z\theta$, then let

$$t_i^{(j+1,k)} = t_i^{(j,k)} \qquad \text{for all } i.$$

If not, then

$$\delta^{(j,k)} = \| t_{i_1}^{(j,k)} - t_{i_2}^{(j,k)} - p^{(j,k)}\theta \|,$$

where $p^{(j,k)} \in Z$ and either $t_{i_1}^{(j,k)}$ or $t_{i_2}^{(j,k)}$ is not in $Z\theta$. Without loss of generality, say $t_{i_1}^{(j,k)} \notin Z\theta$. Let

$$\delta^{(j,k)*} = t_{i_1}^{(j,k)} - t_{i_2}^{(j,k)} - p^{(j,k)}\theta.$$

In this case define

$$t_i^{(j+1,k)} = \begin{cases} t_i^{(j,k)} - \delta^{(j,k)*} & \text{if } t_i^{(j,k)} - t_{i_1}^{(j,k)} \in Z\theta \\ t_i^{(j,k)} & \text{otherwise.} \end{cases}$$

We define $f^{(j+1,k)}$ to have jump discontinuities of size $e(s_i)$ at $-t_i^{(j+1,k)} - p\theta$ for $0 \leqq p < n_k$ where $0 \leqq i < m$. Again it is possible that two or more discontinuities will coincide; in that case we take the total jump at that point to be the product of the appropriate $e(s_i)$'s.

For a fixed j and k, we think of the $t_i^{(j,k)}$ as vertices of a graph in which there is an edge connecting $t_i^{(j,k)}$ and $t_{i'}^{(j,k)}$ whenever their difference is a multiple of θ. Then for each k, the transition from $j \to j+1$ connects at least one additional pair of vertices ($t_{i_1}^{(j,k)}$ and $t_{i_2}^{(j,k)}$), until for some j_k, $\delta^{(j_k,k)}$ is in $Z\theta$, and the process terminates. After at most $m-1$ steps, all the vertices would be connected, so for each k, $j_k \leqq m-1$. We let $j_0 = \max j_k$. By an elementary graph theory argument using the Euler characteristic, the same number of new connections are required to connect all the vertices no matter which order the edges are drawn in. Therefore we have that no (j,k) graph is connected for $j < j_0$. However, it is possible that some or all of the (j_0, k) graphs are connected.

We will show that for each $j < j_0$ the sequence $f^{(j,k)}$ has the following two properties:

$$(6) \qquad |c^{(j,k)}| = \left| \int f^{(j,k)}(x)\,d\mu(x) \right| \to 1 \qquad \text{as } k \to \infty;$$

and if we let $d^{(j,k)}$ = the number of discontinuities of $f^{(j,k)}$, we have that $d^{(j,k)}$ is on the order of n_k, that is

$$d^{(j,k)}/n_k \not\to 0 \qquad \text{as } k \to \infty. \tag{7}$$

We then show that for $j = j_0$, we can find a subsequence $\{k_q\}$, such that $f^{(j_0,k_q)}$ has properties (6) and (7).

Note that $f^{(0,k)}$ does have these properties: Property (6) follows by (2). The condition that no partial sum of the s_i's adds up to an integer implies that the only way that discontinuities which coincide can cancel each other out is to have m coincide. Since at least one $t_i \notin Z\theta$, $f^{(0,k)}$ has at least two columns of discontinuities that do not overlap at all; thus it is impossible for m discontinuities to coincide. Therefore $f^{(0,k)}$ has at least $2n_k$ discontinuities and so satisfies (7).

Now we show that the process for obtaining $f^{(j+1,k)}$ from $f^{(j,k)}$ preserves properties (6) and (7) for $j+1 < j_0$, and for a subsequence of k's, preserves both properties for $j+1 = j_0$.

Property (6): It will be sufficient to show that

$$\lim_{k\to\infty} \|f^{(j,k)} - f^{(j+1,k)}\|_1 = 0, \tag{8}$$

for $j+1 \leqq j_0$ (Property (6) is true even for j_0 without passing to a subsequence). We obtained $f^{(j+1,k)}$ from $f^{(j,k)}$ by moving rn_k points of discontinuity each a distance $\delta^{(j,k)}$, where $r < m$. Thus by the triangle inequality, we need only show that $2rn_k\delta^{(j,k)} \to 0$ as $k \to \infty$, or equivalently that

$$\lim_{k\to\infty} n_k \delta^{(j,k)} = 0. \tag{9}$$

Let $A^{(j,k)} = \{x : |f^{(j,k)}(x) - c^{(j,k)}| < \min |1 - e(\Sigma_{q=1}^{m_0} s_{i_q})|,\ m_0 \leqq m-1\}$, where the minimum is over all partial sums of the s_i's. By hypothesis of the theorem, this minimum is not 0. Since $f^{(j,k)}$ satisfies property (6), we have as in (3) that

$$\lim_{k\to\infty} \mu(A^{(j,k)}) = 1.$$

Since no two adjacent intervals between discontinuities of $f^{(j,k)}$ can both belong to $A^{(j,k)}$ (we assume $f^{(j,k)}$ has at least one of its distances between discontinuities not in $Z\theta$; otherwise $f^{(j+1,k)} = f^{(j,k)}$ and (8) is trivial), we thus conclude that

$$\lim_{k\to\infty} d^{(j,k)} \delta^{(j,k)} = 0.$$

Since $f^{(j,k)}$ satisfies property (7), this gives (9).

Property (7): We have already argued that discontinuities only cancel when m coincide and thus that there are at least $2n_k$ discontinuities in $f^{(j+1,k)}$ unless all the distances between them are in $Z\theta$. We have seen by the graph theory argument above that $f^{(j+1,k)}$ cannot have all distances between discontinuities in $Z\theta$ for any k unless $j+1=j_0$. Therefore we are done except for the case $j+1=j_0$. If there is a subsequence $\{k_q\}$ such that not all the distances between discontinuities of $f^{(j_0,k_q)}$ are in $Z\theta$, we pick this as our subsequence and we are done. So suppose there is not. This means that for k sufficiently large, $j_k=j_0$, so that $\delta^{(j,k)}\notin Z\theta$ for any $j<j_0$, and $p^{(j,k)}$ is defined for $j<j_0$. There are $2|p^{(0,k)}|$ discontinuities of the form $-t_{i_1}^{(0,k)}-p\theta$ in $f^{(j_0,k)}$ that do not coincide with any of the form $-t_{i_2}^{(0,k)}-q\theta$, since once the distance between $t_{i_1}^{(0,k)}$ and $t_{i_2}^{(0,k)}$ is set at $p^{(0,k)}\theta$ in $f^{(1,k)}$, it remains fixed at that distance for all future j. Thus there are at least $2|p^{(0,k)}|$ discontinuities in $f^{(j_0,k)}$ that do not cancel out, and so it is sufficient to find a subsequence $\{k_q\}$ such that $p^{(0,k_q)}$ is on the order of n_{k_q}. We do this by letting k be in $\{k_q\}$ if and only if $|p^{(0,k)}|>|p^{(0,k-1)}|$. Since $\delta^{(0,k)}$ is the minimum distance in $f^{(0,k)}$, this condition insures that $t_{i_1}^{(0,k)}-t_{i_2}^{(0,k)}-p^{(0,k)}\theta$ did not occur as a distance in $f^{(0,k-1)}$ and thus that $|p^{(0,k)}|>n_{k-1}$. Because θ has bounded partial quotients, n_{k-1} is on the order of n_k. We need only show that there is an infinite sequence $\{k_q\}$ such that $|p^{(0,k_q)}|>|p^{(0,k_q-1)}|$. This is true since $t_{i_1}^{(0,k)}-t_{i_2}^{(0,k)}\notin Z\theta$.

Now we have produced a sequence $\{f^{(j_0,k_q)}\}_{q=1}^{\infty}$ that has properties (6) and (7) and has $\delta^{(j_0,k_q)}$ of the form $p^{(j_0,k_q)}\theta$ for all q. For q sufficiently large,

$$|p^{(j_0,k_q)}|\leqq mn_{k_q} \tag{10}$$

since $p^{(j_0,k_q)}$ was obtained from a distance in f by making at most $m-1$ substitutions by multiples of θ with the size of the multiple bounded by n_{k_q}. As before, property (6) implies that

$$\lim_{q\to\infty} d^{(j_0,k_q)}\delta^{(j_0,k_q)}=0$$

and thus by property (7),

$$\lim_{q\to\infty} n_{k_q}\delta^{(j_0,k_q)}=0.$$

But then,

$$\lim_{q\to\infty} n_{kq}\left(\min_{|p|<mnk_q}\|p\theta\|\right)=0,$$

which is impossible for θ with bounded partial quotients by Lemma 2.1. This completes the proof of the theorem.

If we could remove the condition on ratios of successive values, Theorem 3.1 would give necessary and sufficient conditions for an arbitrary step function to be a multiple of a coboundary (bounded partial quotient case). In the case of step functions with two or three points of discontinuity, the ratio condition is always satisfied so we do have this complete answer. The two point case just repeats results of Section 2. The three point case can be used to give necessary and sufficient conditions for f_{t_1,s_1} to be cohomologous to f_{t_2,s_2}. Thus it determines the equivalence classes among the representations of non-type I groups built from these cocycles.

However it is clear that the ratio condition cannot be removed for functions with four or more points of discontinuity. A product of translates of step functions with all interval lenths in $Z\theta$ will not necessarily have all interval lengths in $Z\theta$. One class of easy examples of this are functions of the form $f=(f_{k\theta,s})({}^{r}f_{k\theta,-s})$ where $k\in Z$, $s\in X$, and $r\notin Z\theta$. (Note that Theorem 3.1 does not apply since $e(s)$ and $e(-s)$ occur as ratios of successive values.) As we will have use for the solutions for functions of this form later, we make note of them in the following Lemma:

LEMMA 3.2. *The function* $f_{r,k\theta,s}=(f_{k\theta,s})({}^{r}f_{k\theta,-s})$ *is a coboundary. A solution to* $g(x+\theta)=f_{r,k\theta,s}(x)g(x)$ *is given by* $g=f^{(k)}_{r,-s}$ *for* $k>0$, *and by* $g={}^{-k\theta}f^{(-k)}_{r,s}$ *for* $k<0$.

PROOF. By Corollary 2.3, $f_{k\theta,s}$ has solution $\lambda=e(-s\{k\theta\})$; $g(x)=e(-skx)P_k(x)$. Thus $f_{r,k\theta,s}$ has solution $\lambda=e(-s\{k\theta\})e(s\{k\theta\})=1$;

$$g(x)=(e(-skx))(e(skx))(P_k(x))\overline{({}^{r}P_k(x))}$$

$$=(f_{r,-sk}(x))(P_k(x))\overline{({}^{r}P_k(x))}.$$

Recalling the definition of P_k given in Corollary 2.3, we see that for $k>0$, this g has jump discontinuities of size $e(s)$ at $0,-\theta,\dots,-(k-1)\theta$ and jump discontinuities of size $e(-s)$ at $-r,-r-\theta,\dots,-r-(k-1)\theta$. The lemma for $k>0$ follows by noting that multiples of solutions are still solutions. The case $k<0$ then follows by the fact that $f_{-k\theta,s}=e(s)({}^{-k\theta}f_{k\theta,-s})$.

The examples produced by Corollary 3.2 still have the property that they are formed from products of translates of multiples of coboundaries of the form $f_{k\theta,s}$. In particular, they have the property that at least one of their distances between (not necessarily adjacent) discontinuities is in $Z\theta$. In [16], Veech raised the question of whether any step function which is a multiple of a coboundary must have at least one of its distances between discontinuities in $Z\theta$ (bounded partial quotients case). The following theorem shows that the actual situation is much more complex, and reminiscent of the unbounded partial quotients results of Section 2.

THEOREM 3.3. *Let $t = \sum_{k=0}^{\infty} b_k n_k \theta$ and $r = \sum_{k=0}^{\infty} d_k n_k \theta$ where $|b_k|, |d_k| < a_{k+1}$. (As before, the a_k's are the partial quotients for θ and the n_k's are the denominators.) Suppose*

$$\sum_{k=0}^{\infty} |b_k| \left(\sum_{j=0}^{k} |d_j| n_j \right) \| n_k \theta \| < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} |d_k| \left(\sum_{j=0}^{k} |b_j| n_j \right) \| n_k \theta \| < \infty.$$

Then the function $f_{r,t,s} = (f_{t,s})({}^r f_{t,-s})$ is a coboundary.

PROOF. The method is that of the proof of Theorem 2.5. Accordingly, we define $t_k = \sum_{j=0}^{k} b_j n_j \theta$ and $r_k = \sum_{j=0}^{k} d_j n_j \theta$, and we let

$$f_k = ({}^{t_{k-1}} f_{r, b_k n_k \theta, s}).$$

Then $f_{r,t,s} = \Pi_{b_k \neq 0} f_k$ with the product converging in L^1. We will produce a solution g_k to $g_k(x+\theta) = f_k(x) g_k(x)$, such that $\Pi_{b_k \neq 0} g_k$ converges in L^1. We show convergence by proving $\sum_{b_k \neq 0} \| 1 - g_k \|_1 < \infty$.

We assume without loss of generality that b_k is positive. Using Lemma 3.2 and ignoring translation (since $\| 1 - g_k \|_1$ is independent of translation) we take $g_k = \alpha f_{r,s}^{(b_k n_k)}$, where α is a constant, $|\alpha| = 1$, to be determined later. Now, by the triangle inequality,

$$\sum_{b_k \neq 0} \| 1 - \alpha f_{r,s}^{(b_k n_k)} \|_1 \leqq \sum_{b_k \neq 0} \| 1 - \alpha f_{r_{k-1},s}^{(b_k n_k)} \|_1 + 2 \sum_{b_k \neq 0} |b_k| n_k \left\| \sum_{j=k}^{\infty} d_j n_j \theta \right\|.$$

The second summand is dominated by

$$2 \sum_{k=0}^{\infty} |b_k| n_k \sum_{j=k}^{\infty} |d_j| \| n_j \theta \| = 2 \sum_{j=0}^{\infty} |d_j| \left(\sum_{k=0}^{j} |b_k| n_k \right) \| n_j \theta \|$$

which converges by hypothesis. It remains to show α can be picked so that

$$\sum_{b_k \neq 0} \| 1 - \alpha f_{r_{k-1},s}^{(b_k n_k)} \|_1 < \infty.$$

Recall that (taking into account cancellations) $f_{r_{k-1},s}^{(b_k n_k)}$ has jump discontinuities of size $e(s)$ at $\theta, 2\theta, \ldots, r_{k-1}$; and jump discontinuities of size $e(-s)$ at $(1 - b_k n_k)\theta, (2 - b_k n_k)\theta, \ldots, r_{k-1} - b_k n_k \theta$. Since $|d_j| < a_{j+1}$, we have that $\sum_{j=1}^{k-1} d_j n_j < n_k$ so that these two types of discontinuities interlace. Thus $f_{r_{k-1},s}^{(b_k n_k)}$ has an interval of length $|b_k| \| n_k \theta \|$ followed by one of unknown length, followed by one of length $|b_k|\,\|n_k\theta\|$, and so on; with its value on all the pieces of unknown length the same. Choose α so that $\alpha f^{(b_k n_k)}_{r_{k-1},s}$ is 1 on the intervals of unknown length. Then we are done since $\Sigma_{k=0}^{\infty}|b_k|(\Sigma_{j=0}^{k}|d_j|n_j)\|n_k\theta\|$ converges by hypothesis.

The hypotheses of Theorem 3.3 will be satisfied when either r or t is so rapidly approximable by multiples of θ as to make $f_{r,s}$ or $f_{t,s}$ have a solution by Theorem 2.5. This is as we would expect. For other r and t, the hypotheses can be roughly characterized as requiring that both t and r are rapidly approximable by multiples of θ (though somewhat less so than required by Theorem 2.5), and that the sequences of best approximations for t are nicely related to those for r. We make this more precise by producing a class of r and t that satisfy these hypotheses. Note that if $|b_k|,|d_k|<a_{k+1}$ we have

$$\sum_{k=0}^{\infty}|b_k|\left(\sum_{j=0}^{k}|d_j|n_j\right)\|n_k\theta\|\leqq\sum_{b_k\neq 0}n_{j_k+1}\|n_{k-1}\theta\|,$$

where $j_k=\max\{j\leqq k: d_j\neq 0\}$; and similarly

$$\sum_{k=0}^{\infty}|d_k|\left(\sum_{j=0}^{k}|b_j|n_j\right)\|n_k\theta\|\leqq\sum_{d_k\neq 0}n_{j'_k+1}\|n_{k-1}\theta\|,$$

where $j'_k=\{\max j\leqq k: b_j\neq 0\}$. Thus since $n_k/n_{k+2}\leqq\frac{1}{2}$, $t=\Sigma_{k=0}^{\infty}b_kn_k\theta$ and $r=\Sigma_{k=0}^{\infty}d_kn_k\theta$ will satisfy the hypotheses of Theorem 3.3 whenever $|b_k|,|d_k|<a_{k+1}$. and we have for k sufficiently large that if $b_k\neq 0$, then $d_j=0$ for, say, $2k/3<j<3k/2$. In this case, for nonzero b_k (or d_k), n_{j_k+1} (or $n_{j'_k+1}$) is bounded by $n_{2k/3}$ and so the series in question will be bounded by $\Sigma_{k=0}^{\infty}(\frac{1}{2})^{(k/6)}$. Therefore we have an uncountable class of examples of Theorem 3.3 coming from sparse sequences $\{b_k\}$ and $\{d_k\}$ which mesh sparsely as well. Multiples of θ cannot have representations coming from such sparse sequences unless they are eventually identically 0, since then θ would have a representation coming from an eventually sparse sequence. This is impossible since the size of $\Sigma_{k=k_0}^{\infty}b_kn_k\theta$ cannot be comparable to the size of $\Sigma_{k=0}^{k_0-1}b_kn_k\theta$ if $b_{k_0}=0$ and $|b_k|<a_{k+1}$. Thus these examples give a negative answer to the question of Veech.

Now let $r_0=\Sigma_{k=0}^{\infty}b_kn_k\theta$ where $b_k=1$ if $k=2^{2j}$ for some integer j and $b_k=0$ otherwise. Let $K\subset X$ be defined by $K=\{t: f_{r_0,t,s}$ is a coboundary for all $s\}$. K is uncountable since it includes all t's of the form $t=\Sigma_{k=0}^{\infty}d_kn_k\theta$ where $d_k=0$ if $k\neq 2^{2j+1}$ for some integer j and $d_k=0$ or 1 if $k=2^{2j+1}$. We would like to produce a t in K such that $\int g_{t,r,s}d\mu$ is nonzero for all rational s. We follow a technique used in [14] in the single interval case. If $t_1,t_2\in K$, then t_1-t_2 is in K with solution given by

$$g_{r_0,t_1-t_2,s}={}^{-t_2}(g_{r_0,t_1,s}g_{r_0,t_2,s}).$$

Thus $\int g_{r_0,t_1-t_2,s}d\mu = 0$ if and only if $g_{r_0,t_1,s} \perp g_{r_0,t_2,s}$ in $L^2(X,\mu)$. Since $L^2(X,\mu)$ is separable, if we fix s and fix t_2 in K, there are only countably many t_1 such that $g_{r_0,t_1,s} \perp g_{r_0,t_2,s}$. Thus for each s, there are only countably many t's in $K = K + t_2$ such that $\int g_{r_0,t,s}d\mu \neq 0$. Thus we have an uncountable collection of t's such that $\int g_{r_0,t,s}d\mu \neq 0$ for any rational s. Each such t will have the property that for almost all x, the sequence $\sum_{j=0}^{n-1} \chi_{[0,t)}(x + j\theta) - \chi_{[r_0,r_0+t)}(x + j\theta)$ is not uniformly distributed mod q for any $q > 1$.

Techniques similar to those required to prove Theorem 2.6 can be used to produce a partial converse to Theorem 3.3: for $s = \frac{1}{2}$, if $f_{r,t,s}$ is a multiple of a coboundary, then r and t have representations $r = \sum_{k=0}^{\infty} b_k n_k \theta$ and $t = \sum_{k=0}^{\infty} d_k n_k \theta$ such that

$$\lim_{k\to\infty}\left(n_k \sum_{j=k}^{\infty} |b_j| \, \|n_j\theta\|\right)\left(n_k \sum_{j=k}^{\infty} |d_j| \, \|n_j\theta\|\right) = 0.$$

However, as in the unbounded partial quotients case in the previous section, a class of r and t remain for which it is undetermined whether $f_{r,t,s}$ is a multiple of a coboundary.

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